fifth place.[1] Below 5° it gives the result correct to five places in about 60 per cent. of all cases and shows an error of one in the fifth place in about 40 per cent. of cases.

Below 2°, logarithmic interpolation is certainly preferable to ordinary interpolation for three reasons: (1) it gives the correct result in a larger number of cases; (2) it gives an error in the fifth place in a smaller number of cases; (3) it never gives an error greater than one in the fifth place.

Above 2°, ordinary interpolation is preferable for the same three reasons. All cases between 1° 30′ and 2° 15′ were examined.

An examination of 200 cases in the neighborhood of 84° shows that ordinary interpolation will give the result correct to five places in about 83 per cent., and will show an error of one in the fifth place in 17 per cent. of cases.

Below is a table of results of the statistical investigation.

Interval of Angle.	Number of Cases Examined.	Per Cent. of Cases Giving Result Correct to 5 Places.		Per Cent. of Cases Showing an Error of One in the 5th Place.		Per Cent. of Cases Showing an Error of Two or More in the Fifth Place.	
		Ordinary.	Logarithmic.	Ordinary.	Logarithmic.	Ordinary.	Logarithmic.[2]
1′–1°	45	Fails	60	Large	40		0
1° –1° 30′	97	33		57		10	0
1° 30′–1° 45′	All	55	63	45	37	0	0
1° 45′–2°	All	63	69	37	31	0	0
2° –2° 15′	All	71	62	29	38	0	0
2° 15′–3°	55	78	60	22	40	0	0
3° –4°	55	80		20		0	0
4° –5°						0	0
5° –5° 45′	All		66		34	0	0
5° 45′–6°	All		64		36		[3]

A THEOREM ABOUT ISOGONAL CONJUGATES.

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As an introduction let us recall two well-known theorems of elementary plane geometry:

THEOREM I. *Given a triangle* $A_1A_2A_3$ *(Fig. 1) and any point P not a vertex, join P to the three vertices and reflect each of the three lines thus drawn in the bisector of the angle at the corresponding vertex. The three reflected lines will meet in a point P′, called the isogonal conjugate of P.*

THEOREM II. *Given a triangle and a point marked at random on each side. If three circles be drawn, one through each vertex and the two adjacent marked points, these three circles meet in a common point.*

[1] The first error of two in the fifth place enters at 5° 53′.7

[2] All cases examined for this column.

[3] At 5° 53′.7 the first error of 2 in the fifth place enters.

We propose to prove a new theorem:

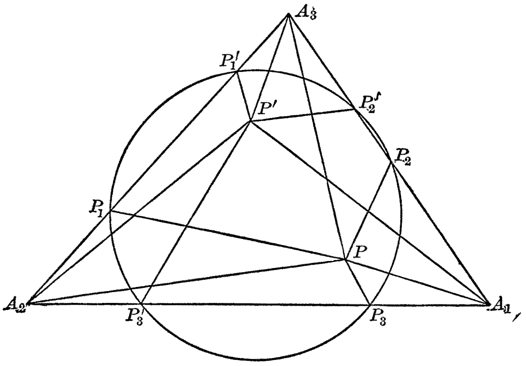

FIG. 1.

THEOREM III. *Given a triangle* $A_1A_2A_3$ *(Fig. 2) and a circle cutting each side in two points,* P_1, P_1'; P_2, P_2'; P_3, P_3', *where* P_i *and* P_i' *are on the side opposite* A_i. *If a point* P *be located as the common intersection of the three circles through the three sets of points* $P_iA_jP_k$; *and a point* P' *as the common intersection of three circles through* $P_i'A_jP_k'$; *then* P *and* P' *are isogonal conjugates.*

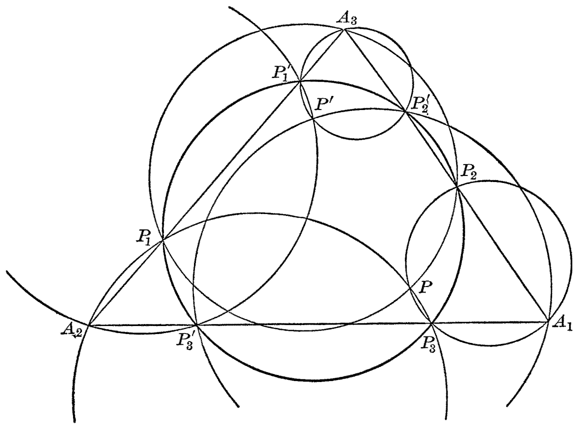

FIG. 2.

Lemma. Let P and P' be isogonal conjugates with regard to the triangle $A_1A_2A_3$ (Fig. 1).

Choose three points P_1, P_2, P_3, one on each side of the triangle, so that

$$\measuredangle PP_1A_2 = \measuredangle PP_2A_3 = \measuredangle PP_3A_1 = \theta$$

and choose three other points P_1', P_2', P_3', one on each side, and such that

$$\measuredangle P'P_1'A_3 = \measuredangle P'P_2'A_1 = \measuredangle P'P_3'A_2 = \theta$$

where θ is any angle. Then we can show that these six chosen points all lie on a circle.

From the similar triangles PA_1P_3 and $P'A_1P_2'$

$$\frac{A_1P_3}{A_1P} = \frac{A_1P_2'}{A_1P'}$$

and from similar triangles PA_1P_2 and $P'A_1P_3'$

$$\frac{A_1P_3'}{A_1P'} = \frac{A_1P_2}{A_1P}$$

Multiply these two equations together, canceling denominators.

$$A_1P_3 \times A_1P_3' = A_1P_2 \times A_1P_2'.$$

Hence P_2, P_2', P_3 and P_3' are four concyclic points. In like manner P_1, P_1', P_2, P_2', and also P_3, P_3', P_1, P_1' are concyclic. This gives us, apparently, three circles on each of which lie four of the six points. Now the sides of the triangle are the three radical axes of these circles taken two at a time. All three circles cannot be distinct because these three radical axes do not concur in a point. Hence at least two circles coincide, and so all six points are concyclic.

Observe also that by the construction of Fig. 1

$$\measuredangle PP_iA_j + \measuredangle PP_kA_j = 180°$$

and so P, P_i, A_j, P_k are concyclic. Therefore P is the point where the three circles $P_iA_jP_k$ meet. Similarly P' is the intersection of the three circles $P_i'A_jP_k'$.

We get the proof of Theorem III as follows:

Suppose we start with the triangle $A_1A_2A_3$; put down a circle cutting its sides in points P_1, P_1'; P_2, P_2'; P_3, P_3'; and locate P as the point common to the three circles $P_iA_jP_k$. Then the lemma shows that the isogonal conjugate of P lies on each of the three circles $P_i'A_jP_k'$; and since only one point does this, then the isogonal conjugate of P is that point.

If we interchange two of the points that lie an a side, say P_1 and P_1', we get a new pair of isogonal conjugates. Evidently four different pairs of isogonal conjugates can be obtained in this way, all determined by the same given circle cutting the sides. If each pair of isogonal conjugates be connected by a straight line and the perpendicular bisectors of these four lines be erected; then these four bisectors all meet in the center of the given circle. The only proof of this, which I have been able to devise, is rather tedious; but doubtless a simple and elegant proof exists.

If the given circle is tangent to one side of the triangle, but cuts the other two in distinct points, then it determines only two pairs of isogonal conjugates. If it be tangent to two sides it determines but one pair of isogonal conjugates. If it be an inscribed or escribed circle it determines one self-conjugate point, its center.